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2002 J. Phys. A: Math. Gen. 35 6695

(http://iopscience.iop.org/0305-4470/35/31/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6695–6712 PII: S0305-4470(02)33284-0

Decoherence in infinite quantum systems

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Received 31 January 2002, in final form 13 June 2002
Published 26 July 2002
Online at stacks.iop.org/JPhysA/35/6695

Abstract
Environmentally induced classical properties in quantum spin systems in the thermodynamic limit are discussed. In particular, it is shown that such a system, subjected to a specific interaction with another quantum system, may be described for all practical purposes as a classical dynamical system. This provides an alternative approach to the classical $\hbar \to 0$ limit for the emergence of reversible classical evolution from dissipative quantum dynamics.

PACS numbers: 03.65.Ta, 03.65.Yz

1. Introduction

In recent years, decoherence has been widely discussed and accepted as the mechanism responsible for the appearance of classicality in quantum measurements and the absence, in the real world, of Schrödinger-cat-like states [1–4]. The basic idea behind it is that classicality is an emergent property induced in quantum open systems by their environment. It is marked by the dynamical transition of a vast majority of pure states of the system to statistical mixtures. In other words, decoherence is a process of a continuous interaction between the system and its environment (sometimes called a continuous fuzzy measurement [5]), which results in limiting the validity of the superposition principle in the Hilbert space of the system. This resolves the measurement problem essentially in the following way: since any realistic measuring apparatus is macroscopic, it necessarily interacts with its environment and so, almost instantaneously after the measurement, the reduced state of the measuring instrument is, for all practical purposes, indistinguishable from a state representing a classical probability distribution over determined but unknown values of the measured observable. Information required to exhibit quantum interference effects between distinct pointer states is immediately lost in the external degrees of freedom.

In order to study decoherence, the analysis of the evolution of the reduced density matrices obtained by tracing out the environmental variables is the most convenient strategy. For a large class of physical phenomena, this evolution can be described by a dynamical semigroup, whose generator is given by a Markovian master equation. The loss of quantum coherence in the Markovian regime was established in a number of open systems [6, 7] giving clear evidence of environment-induced superselection rules. In a recent paper [8] a thorough analysis of the superselection structure induced by a dynamical semigroup, which is also contractive in the trace norm, was presented. It was achieved by the use of the isometric-sweeping decomposition, which singles out a subalgebra, $\mathcal{M}_1$, of the algebra of all observables whose elements are immune to the process of decoherence and so evolve in a unitary way according to Schrödinger dynamics in the Heisenberg picture. Other observables decay in time (in the appropriate topology) to elements in $\mathcal{M}_1$. It means that expectation values of those observables, which are 'orthogonal' to the algebra $\mathcal{M}_1$, tend to zero when time goes to infinity. Hence, after the so-called decoherence time, the existence of such observables cannot be experimentally detected. Therefore, when decoherence happens almost instantaneously then the subalgebra $\mathcal{M}_1$ represents effective observables of the quantum system [9]. It is said that environmentally induced superselection rules appear, when the centre $Z(\mathcal{M}_1)$ of the algebra $\mathcal{M}_1$ is non-trivial. In a particular case, when decoherence affects all but a subset of the so-called pointer states, the algebra of effective observables becomes commutative, isomorphic to the algebra of bounded sequences $l^\infty(\Gamma)$, where Γ is a finite or infinite discrete set. And, as was shown in [9], this is the only possible Abelian subalgebra, which can be induced by environmental decoherence. Because on the algebra $l^\infty(\Gamma)$ there are no non-trivial derivations so the evolution, when restricted to $\mathcal{M}_1$, must be trivial. Hence, the above scheme, although fruitful in the discussion of quantum measurements and the absence of Schrödinger-like-cat states, cannot be used for derivation of time continuous classical dynamics.

However, it should be noted that those results concern only quantum systems with a finite number of degrees of freedom, whose observables, due to the Stone–von Neumann uniqueness theorem, form a factor algebra of type I. A new perspective is obtained when we pass to the thermodynamic limit. Since in such a case the von Neumann algebra of observables is a continuous factor [10] so it contains continuous Abelian subalgebras as well. In the case when the algebra $\mathcal{M}_1$ of effective observables is indeed a continuous Abelian algebra $L^\infty(\Omega)$, the evolution restricted to $\mathcal{M}_1$ induces a time continuous flow S_t on the configuration space Ω. Hence, in such a case the quantum system can be thought of, for all practical purposes, as a classical one whose evolution is given by a trajectory $x \to S_t x$ in Ω. More precisely, $\mathcal{M}_1$ is a Hilbert space representation of some classical system and the group of automorphisms $T_t|_{\mathcal{M}_1}$ represents a linear equation of motion of classical statistical mechanics, which is induced by a 'nonlinear' equation of motion of classical point mechanics. This constitutes an alternative approach (to the classical limit of Wigner functions [11]) for generating classical deterministic dynamics from dissipative quantum systems. The main objective of this paper is to study decoherence and its properties (in the Markovian approximation) on a matricial factor of type II_1 representing an infinite spin system. As a particular case we present a semigroup, for which the algebra of effective observables is just the algebra of functions on a circle, and whose evolution is induced by a rotation of the circle.

2. Spin system on a lattice

For infinite systems, it has been argued by Haag and Kastler [12] that the algebra of observables $\mathcal{A}$ of the system has a quasi-local structure in the following sense. There exists a set $\mathcal{F}$ of bounded regions $\Lambda \subset \mathbb{R}^3$ such that $\bigcup_{\mathcal{F}} \Lambda = \mathbb{R}^3$, for $\Lambda_1, \Lambda_2 \in \mathcal{F}$ there exists $\Lambda \in \mathcal{F}$ with $\Lambda_1 \cup \Lambda_2 \subset \Lambda$, and for every $\Lambda \in \mathcal{F}$ there exists $\Lambda' \in \mathcal{F}$ such that $\Lambda \cap \Lambda' = \emptyset$. Moreover, for any $\Lambda \in \mathcal{F}$ there is a C^*-algebra $\mathcal{A}_\Lambda$ with unit satisfying

$$\Lambda_1 \subset \Lambda_2 \quad \Rightarrow \quad \mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$$
$$\Lambda_1 \cap \Lambda_2 = \emptyset \quad \Rightarrow \quad \left[\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}\right] = 0$$
$$\bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda \text{ is norm dense in } \mathcal{A}.$$

Suppose now that our infinite system consists of spin-$\frac{1}{2}$ particles located at each point of a discrete lattice $\Gamma \subset \mathbb{R}^3$. We assume that Γ is both countable and infinite. Let $\mathcal{F}$ denote the set of all subsets Λ of Γ such that $|\Lambda| < \infty$, where $|\Lambda|$ denotes the number of points in Λ. Then, for any point $x \in \Gamma$, the algebra $\mathcal{A}_{\{x\}}$ is just an algebra of 2×2 matrices, and so $\mathcal{A}_\Lambda$ is a full algebra of $2^{|\Lambda|} \times 2^{|\Lambda|}$ matrices for any $\Lambda \in \mathcal{F}$. The quasi-local algebra $\mathcal{A}$ is given by the direct limit of $\mathcal{A}_\Lambda$, $\Lambda \in \mathcal{F}$. In order to enter the standard description of quantum mechanics, we take the GNS representation π_ω of $\mathcal{A}$ with respect to a faithful factor state ω on $\mathcal{A}$ [13]. In this paper, we consider only those factor states ω such that $\mathcal{M} = \pi_\omega(\mathcal{A})''$, the bicommutant of $\pi_\omega(\mathcal{A})$, is a finite von Neumann algebra acting on the Hilbert space $\mathcal{H}_\omega$. Clearly, $\mathcal{M}$ is a factor of type II_1 in this case. Typically, such states are of the form

$$\omega(A) = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta H} A)}{\mathrm{Tr}(\mathrm{e}^{-\beta H})}$$

where $H = H^* \in \mathcal{A}$ and Tr denotes a unique normalized trace on $\mathcal{A}$. More generally, we may take states which are quasi-equivalent to the tracial state Tr (a state ω_1 is quasi-equivalent to a state ω_2 if every π_1 normal state is π_2 normal and vice versa, where π_1 and π_2 denote the GNS representations associated with ω_1 and ω_2 respectively [13]).

The above description defines the kinematics of the quantum spin system. We next consider its dynamics in the case when it interacts with an environment. Suppose that environmental observables constitute an algebra $\mathcal{B}$ of operators acting on a Hilbert space $\mathcal{H}_E$. The reduced dynamics of $\pi_\omega(\mathcal{A})$ is given by the conditional expectation of the Schrödinger-type dynamics (in the Heisenberg picture) of the joint algebra $\pi_\omega(\mathcal{A}) \otimes \mathcal{B}$, where the tensor product is defined on the Hilbert space $\mathcal{H}_\omega \otimes \mathcal{H}_E$. However, it should be noted that in general the unitary evolution of the joint system does not exist on the algebra $\pi_\omega(\mathcal{A}) \otimes \mathcal{B}$, but rather on its weak closure [14]. Therefore, also the reduced dynamics may be properly defined only on the von Neumann algebra $\mathcal{M} = \pi_\omega(\mathcal{A})''$, which we call the algebra of (contextual) observables of the system [15]. Clearly, it contains operators representing quasi-local observables and also many others. In general, the reduced dynamics being the composition of a conditional expectation with a unitary evolution is represented by a family of completely positive superoperators on $\mathcal{M}$. As was mentioned in the introduction, we restrict our considerations to the Markovian regime, and so assume that the evolution of the algebra $\mathcal{M}$ is given by a dynamical semigroup. By a quantum dynamical semigroup we mean a weakly* continuous semigroup T_t, $t \geqslant 0$, of completely positive and normal maps on a von Neumann algebra $\mathcal{M}$ such that for all $t \geqslant 0$, T_t is contractive in the operator norm and $T_t(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1}$ denotes the identity operator in $\mathcal{M}$ [16]. With an additional assumption that T_t is also contractive in the trace norm, we discuss in the next section environmentally induced decoherence in the algebra $\mathcal{M}$ of the spin system.

3. Decoherence in the spin system

Since in this section we present mathematical results concerning the decomposition of $\mathcal{M}$ with respect to the asymptotic properties of a dynamical semigroup T_t we keep the required assumptions at a minimum. Because generalization to the continuous case is straightforward, we consider for simplicity a discrete semigroup T^n, $n \in \{0, 1, 2, \ldots\}$, generated by a single operator $T : \mathcal{M} \to \mathcal{M}$.

Suppose that $\mathcal{M}$ is a factor of type II_1 (not necessarily matricial) acting in a separable Hilbert space $\mathcal{H}$. Let $(\mathcal{H}, \mathcal{M}, D)$ be a *gauge* space, where D is the normalized dimension function [17]. Let Tr be a trace on the algebra $\mathcal{M}$ with the property $\mathrm{Tr}\, P = D(P)$ for any projection $P \in \mathcal{M}$. One can consider the sets $L^1(\mathcal{M})$ and $L^2(\mathcal{M})$ of all operators measurable with respect to the algebra $\mathcal{M}$, which are integrable and square-integrable respectively. The following inclusions of the Banach spaces hold true,

$$\mathcal{M} \subset L^2(\mathcal{M}) \subset L^1(\mathcal{M})$$

and the corresponding norms satisfy $\|\cdot\|_\infty \geqslant \|\cdot\|_2 \geqslant \|\cdot\|_1$. The subspace $\mathcal{M}$ is dense in $L^2(\mathcal{M})$ and $L^1(\mathcal{M})$ in the appropriate norm. Moreover, one has $\mathcal{M}_* = L^1(\mathcal{M})$, where $\mathcal{M}_*$ denotes the pre-dual space of the algebra $\mathcal{M}$. The pairing between operators and states is given by the bilinear form,

$$\mathcal{M} \times L^1(\mathcal{M}) \ni (x, \varphi) \to \mathrm{Tr}(x\varphi) \in \mathbf{C}$$

where by the same symbol Tr we denote the extension of the trace to the space $L^1(\mathcal{M})$. The space $L^1(\mathcal{M})$ is provided with a continuous *-operation determined by the involution of $\mathcal{M}$ in a unique way. More precisely, for any $\varphi \in L^1(\mathcal{M})$, the element φ^* coincides with the ordinary adjoint of φ in the Hilbert space $\mathcal{H}$. It is clear that $\|\varphi^*\|_1 = \|\varphi\|_1$.

Suppose now that the operator $T : \mathcal{M} \to \mathcal{M}$ satisfies

(a) T is two-positive,
(b) T is contractive both in $\|\cdot\|_\infty$ and $\|\cdot\|_1$-norm,
(c) $T(\mathbf{1}) = \mathbf{1}$.

Then T may be extended to a contraction $\overline{T}$ on the space $L^1(\mathcal{M})$. The dual operator $\overline{T}^* : \mathcal{M} \to \mathcal{M}$ is also contractive in both $\|\cdot\|_\infty$ and $\|\cdot\|_1$ norm. Therefore, it has the extension $\overline{\overline{T}^*}$ on $L^1(\mathcal{M})$. By interpolation, the restrictions of $\overline{T}$ and $\overline{\overline{T}^*}$ to $L^2(\mathcal{M})$ are also contractions. To simplify notation, we denote all these operators by T and T^*, i.e.

$$T, T^* : \mathcal{M} \to \mathcal{M} \qquad L^1(\mathcal{M}) \to L^1(\mathcal{M}) \qquad L^2(\mathcal{M}) \to L^2(\mathcal{M}).$$

Clearly T and T^* are Hermitian conjugate with respect to the scalar product in $L^2(\mathcal{M})$. Moreover, $T^* : \mathcal{M} \to \mathcal{M}$ is two-positive and normal.

Proposition 1. $T^*(\mathbf{1}) = \mathbf{1}$ *and so T is a trace preserving operator on $\mathcal{M}$. Moreover, T is normal.*

Proof. $\|T^*(\mathbf{1})\|_2^2 \leqslant \|\mathbf{1}\|_2^2$. So $\|T^*(\mathbf{1}) - \mathbf{1}\|_2^2 = \|T^*(\mathbf{1})\|_2^2 - 2\mathrm{Tr}\, T^*(\mathbf{1})\mathbf{1} + \mathrm{Tr}\mathbf{1} = \|T^*(\mathbf{1})\|_2^2 - \|\mathbf{1}\|_2^2 \leqslant 0$. Hence $\|T^*(\mathbf{1}) - \mathbf{1}\|_2 = 0$. It follows that for any $x \in \mathcal{M}$, $\mathrm{Tr}\, Tx = \mathrm{Tr}\, xT^*(\mathbf{1}) = \mathrm{Tr}\, x$. Finally, by direct calculations, we check that for any $A, B \in \mathcal{M}$ the equality $\mathrm{Tr}[A(\overline{\overline{T}^*})^* B] = \mathrm{Tr}[A(TB)]$ holds true. Since $\mathcal{M}$ is norm dense in $\mathcal{M}_*$ so $(\overline{\overline{T}^*})^* = T$, and hence T is normal. □

Such bi-contractive operators on semifinite von Neumann algebras were studied by Yeadon [18] and, in the case of arbitrary σ-finite von Neumann algebras, by Groh and Kümmerer [19]. For example, the extensions of operator T and its dual to all $L^p(\mathcal{M})$ spaces were given in [18]. The mean ergodic properties of the semigroup of such operators were also discussed in those papers.

Because T is a contraction on Hilbert space $L^2(\mathcal{M})$ so we may define a unitary subspace K for T by

$$K = \{x \in L^2(\mathcal{M}) : \|T^n x\|_2 = \|T^{*n} x\|_2 = \|x\|_2 \ \text{for all}\ n \in \mathbb{N}\}.$$

Since $x \in K$ iff $T^n T^{*n} x = T^{*n} T^n x = x$ for any $n \in \mathbb{N}$ so K is a linear subspace in $L^2(\mathcal{M})$. Both K and its orthogonal complement $K^\perp$ are T and T^* invariant. Moreover, for $x \in K^\perp$ we have

$$w - \lim_{n\to\infty} T^n x = w - \lim_{n\to\infty} T^{*n} x = 0.$$

Proposition 2.

(a) If $x \in K$, then $x^* \in K$,
(b) If $x = x^* \in K$, then $|x|, x^+, x^- \in K$,
(c) $\mathbf{1} \in K$.

Proof. Points (a) and (c) are clear. Suppose that $x = x^* \in K$. Then $-|x| \leqslant x \leqslant |x|$ and so $-T^n|x| \leqslant T^n x \leqslant T^n|x|$. Hence

$$\|T^n x\|_2 \leqslant \|T^n|x|\|_2 \leqslant \||x|\|_2 = \|x\|_2.$$

Because $\|x\|_2 = \|T^n x\|_2$ so, for any $n \in \mathbb{N}$, $\||x|\|_2 = \|T^n|x|\|_2$, and similarly for T^{*n}. Thus $|x| \in K$. Because $2x^+ = |x| + x$ and $2x^- = |x| - x$ so the proof is finished. □

Proposition 3. *Suppose that $x \in K$ and $x \geqslant 0$. Then $E(B) \in K$ for any $B \in \mathcal{B}[0,\infty)$, where $\mathrm{d}E(\lambda)$ is a spectral measure for x, i.e. $x = \int_0^\infty \lambda\, \mathrm{d}E(\lambda)$, and $\mathcal{B}[0,\infty)$ is the σ-algebra of Borel subsets in $[0,\infty)$.*

Proof. Suppose that $n \in \mathbb{N}$. Let us define

$$A_n = \tfrac{1}{2}[n(x-a)^+ + \mathbf{1} - |n(x-a)^+ - \mathbf{1}|]$$

where $a > 0$. Clearly, $A_n \in K$. However, $(x-a)^+ = \int_a^\infty (\lambda - a)\, \mathrm{d}E(\lambda)$ and so

$$n(x-a)^+ - \mathbf{1} = \int_a^\infty n(\lambda - a)\, \mathrm{d}E(\lambda) - \int_0^a \mathrm{d}E(\lambda) - \int_a^\infty \mathrm{d}E(\lambda)$$

where $\int_a^b$ denotes $\int_{[a,b)}$. Therefore,

$$\begin{aligned} |n(x-a)^+ - \mathbf{1}| &= \int_0^a \mathrm{d}E(\lambda) + \int_a^\infty |n(\lambda-a) - 1|\, \mathrm{d}E(\lambda) \\ &= \int_0^a \mathrm{d}E(\lambda) + \int_{a+\frac{1}{n}}^\infty [n(\lambda-a) - 1]\, \mathrm{d}E(\lambda) - \int_a^{a+\frac{1}{n}} [n(\lambda-a) - 1]\, \mathrm{d}E(\lambda) \end{aligned}$$

and so

$$\begin{aligned} 2A_n = {} & \int_a^\infty n(\lambda-a)\, \mathrm{d}E(\lambda) + \int_a^\infty \mathrm{d}E(\lambda) - \int_{a+\frac{1}{n}}^\infty [n(\lambda-a) - 1]\, \mathrm{d}E(\lambda) \\ & + \int_a^{a+\frac{1}{n}} [n(\lambda-a) - 1]\, \mathrm{d}E(\lambda) \\ = {} & \int_a^{a+\frac{1}{n}} n(\lambda-a)\, \mathrm{d}E(\lambda) + \int_{a+\frac{1}{n}}^\infty n(\lambda-a)\, \mathrm{d}E(\lambda) + \int_a^\infty \mathrm{d}E(\lambda) \\ & - \int_{a+\frac{1}{n}}^\infty n(\lambda-a)\, \mathrm{d}E(\lambda) + \int_{a+\frac{1}{n}}^\infty \mathrm{d}E(\lambda) + \int_a^{a+\frac{1}{n}} n(\lambda-a)\, \mathrm{d}E(\lambda) - \int_a^{a+\frac{1}{n}} \mathrm{d}E(\lambda). \end{aligned}$$

Hence

$$A_n = \int_a^{a+\frac{1}{n}} n(\lambda-a)\, \mathrm{d}E(\lambda) + E\left(\left[a + \frac{1}{n}, \infty\right)\right).$$

Now we show that in $L^2(\mathcal{M})$

$$\lim_{n\to\infty}\int_a^{a+\frac{1}{n}} n(\lambda-a)\,\mathrm{d}E(\lambda)=0.$$

Indeed

$$\left\|\int_a^{a+\frac{1}{n}} n(\lambda-a)\,\mathrm{d}E(\lambda)\right\|_2^2 = n^2\int_a^{a+\frac{1}{n}}(\lambda-a)^2\,\mathrm{d}\,\mathrm{Tr}\,E(\lambda)$$
$$\leqslant n^2\frac{1}{n^2}\mathrm{Tr}\,E\left(\left(a,a+\frac{1}{n}\right)\right)=\mathrm{Tr}\,E\left(\left(a,a+\frac{1}{n}\right)\right).$$

Because $\mathrm{d}\,\mathrm{Tr}\,E(\lambda)$ is a probability measure so

$$\lim_{n\to\infty}\mathrm{Tr}\,E\left(\left(a,a+\frac{1}{n}\right)\right)=0.$$

In the same way, we show that $E\left(\left[a+\frac{1}{n},\infty\right)\right)\to E((a,\infty))$ in $L^2(\mathcal{M})$. So $\lim_{n\to\infty}A_n = E((a,\infty))\in K$. Because $\mathbf{1}\in K$ so $E([0,a])\in K$ and, consequently, $E((a,b))\in K$ for any $0\leqslant a<b$. Suppose now that $\mathcal{R}$ is a family of subsets $B\subset[0,\infty)$ such that $E(B)\in K$. Clearly, $\mathcal{R}$ is a σ-algebra containing all open intervals. Hence $\mathcal{B}[0,\infty)\subset\mathcal{R}$. □

Remark. If $x=x^*\in K$ and $x=\int_{-\infty}^{+\infty}\lambda\,\mathrm{d}E(\lambda)$ then $E(B)\in K$ for any $B\in\mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ stands for the Borel σ-algebra.

Let P denote the orthogonal projection from $L^2(\mathcal{M})$ onto K. It is clear that $(Px)^*=P(x^*)$.

Proposition 4. *Suppose that $x\in\mathcal{M}$ and $x\geqslant 0$. Then $Px\geqslant 0$ and $\|Px\|_1=\|x\|_1$.*

Proof. Because $x\geqslant 0$ so $(Px)^*=Px$. Hence $Px=\int_{-\infty}^{\infty}\lambda\,\mathrm{d}E(\lambda)$. Let $P^\perp$ be orthogonal projection onto $K^\perp$. Because for any $B\in\mathcal{B}(\mathbb{R})$, $E(B)\in K$ so

$$\mathrm{Tr}\,E(B)P^\perp x=0=\mathrm{Tr}\,E(B)(x-Px)=\mathrm{Tr}\,E(B)x-\mathrm{Tr}\int_B\lambda\,\mathrm{d}E(\lambda).$$

Hence $\mathrm{Tr}\int_B\lambda\,\mathrm{d}E(\lambda)=\int_B\lambda\,\mathrm{d}\,\mathrm{Tr}\,E(\lambda)\geqslant 0$ which implies that $E((-\infty,0))=0$. Thus $Px\geqslant 0$. Moreover, since $\mathbf{1}\in K$ so $\mathrm{Tr}\,\mathbf{1}P^\perp x=0$. Hence $\mathrm{Tr}\,x=\mathrm{Tr}\,Px$. □

Corollary 5. *P extends to a bounded projection on $L^1(\mathcal{M})$. Its dual P^* is a bounded and positive projection on $\mathcal{M}$.*

Proposition 6. $P^*=P|_{\mathcal{M}}$.

Proof. For $A\in\mathcal{M}$ and $x\in L^2(\mathcal{M})$ we have

$$\mathrm{Tr}\,P^*(A^*)x=\mathrm{Tr}\,A^*Px=\langle A,Px\rangle_{L^2}=\langle PA,x\rangle_{L^2}=\mathrm{Tr}\,(P(A))^*x=\mathrm{Tr}P(A^*)x.$$

Because $L^2(\mathcal{M})$ is dense in $L^1(\mathcal{M})$ so $P^*(A^*)=P(A^*)$ for any $A\in\mathcal{M}$ and thus $P^*=P$ on $\mathcal{M}$. □

Let us define $\mathcal{M}_1=\mathcal{M}\cap K$ and $\mathcal{M}_2=\mathcal{M}\cap K^\perp$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are Banach subspaces of $\mathcal{M}$ and $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$. Moreover, $T=T_1\oplus T_2$, where $T_1=T|_{\mathcal{M}_1}$ and $T_2=T|_{\mathcal{M}_2}$.

Theorem 7. *$\mathcal{M}_1$ is a von Neumann algebra. $T|_{\mathcal{M}_1}$ is a trace preserving *-automorphism of the algebra $\mathcal{M}_1$ with $\left(T|_{\mathcal{M}_1}\right)^{-1}=T^*|_{\mathcal{M}_1}$. On the other hand, for any $A\in\mathcal{M}_2$ and all $\varphi\in L^1(\mathcal{M})$ we have*

$$\lim_{n\to\infty}\mathrm{Tr}\,T^n(A)\varphi=0.$$

Proof. First we prove that if $A, B \in \mathcal{M}_1$, then also $A \cdot B \in \mathcal{M}_1$. Clearly, it suffices to consider only Hermitian A and B. But then

$$2A \cdot B = (A+B)^2 - A^2 - B^2 + \mathrm{i}[(A-\mathrm{i}B)^* \cdot (A-\mathrm{i}B) - A^2 - B^2].$$

Hence it is enough to check that if $x \in \mathcal{M}_1$, then $x^*x \in \mathcal{M}_1$. Since both T and T^* are two-positive so, by the Schwarz inequality,

$$T^{*n}T^n(x^*x) \geqslant (T^{*n}T^n x)^* T^{*n}T^n x = x^*x.$$

Hence $\|x^*x\|_2 \leqslant \|T^{*n}T^n(x^*x)\|_2 \leqslant \|x^*x\|_2$, and so $x^*x \in K$. However, $x^*x \in \mathcal{M}$ so $x^*x \in \mathcal{M}_1$. Thus $\mathcal{M}_1$ is an *-algebra containing the identity operator. Suppose now that $A_\alpha \to A$ in the σ-weak topology. Because both T and T^* are normal so

$$T^{*n}T^n(A) = \lim_\alpha T^{*n}T^n(A_\alpha) = \lim_\alpha A_\alpha = A$$

and similarly for T^nT^{*n}. Hence $A \in \mathcal{M}_1$ and so $\mathcal{M}_1$ is a von Neumann algebra.

Next, we prove properties of the restriction of the operator T to the algebra $\mathcal{M}_1$. Because, by proposition 1, T is trace preserving so it suffices to check only that $T(xy) = T(x)T(y)$ for any $x, y \in \mathcal{M}_1$. Because T satisfies the Schwarz inequality $(Tx)^*Tx \leqslant T(x^*x)$, so for any $x \in \mathcal{M}_1$,

$$\mathrm{Tr}[T(x^*x) - (Tx)^*Tx] = \|x\|_2^2 - \|Tx\|_2^2 = 0.$$

Hence $(Tx)^*Tx = T(x^*x)$ since the state Tr is faithful. Let us define a positive sesquilinear form

$$b_\varphi = \varphi[T(x^*y) - (Tx)^*Ty]$$

where φ is a positive normal state on $\mathcal{M}$. Then $b_\varphi(x,x) = 0$ implies that $b_\varphi(x,y) = 0$ for any $y \in \mathcal{M}_1$. Since φ was arbitrary so $T(x^*y) = T(x^*)T(y)$ for any $x, y \in \mathcal{M}_1$. Because for any $x \in \mathcal{M}_1$, $T^*T(x) = TT^*x = x$ so T is invertible with $\left(T|_{\mathcal{M}_1}\right)^{-1} = T^*|_{\mathcal{M}_1}$. Hence $T|_{\mathcal{M}_1}$ is an *-automorphism of the algebra $\mathcal{M}_1$.

To prove the last statement of this theorem, suppose that $\varphi \in L^2(\mathcal{M})$. Because $A \in K^\perp$ so the assertion follows from the definition. Because $L^2(\mathcal{M})$ is dense in $L^1(\mathcal{M})$ in the trace norm so for $\varphi \in L^1(\mathcal{M})$ and any $\varepsilon > 0$ we may find $x \in L^2(\mathcal{M})$ such that $\|\varphi - x\|_1 < \varepsilon$. Then

$$|\mathrm{Tr}\, T^n(A)\varphi| \leqslant |\mathrm{Tr}\, T^n(A)(\varphi - x)| + |\mathrm{Tr}\, T^n(A)x| \leqslant \varepsilon\|A\| + |\mathrm{Tr}\, T^n(A)x|$$

and so $\lim_{n\to\infty}|\mathrm{Tr}\, T^n(A)\varphi| \leqslant \varepsilon\|A\|$. Since ε was arbitrary, the proof is finished. □

Clearly $\mathcal{M}_1 = \mathrm{Range}\, P|_{\mathcal{M}}$. The algebra $\mathcal{M}_1$ will be called the subalgebra of effective observables or simply effective subalgebra. Next, we describe properties of the projection $P|_{\mathcal{M}}$, which we also denote by P.

Theorem 8. *$P : \mathcal{M} \to \mathcal{M}$ is a completely positive normal norm-1 projection. Moreover,* $\|P\|_{1,1} = 1$.

Proof. At first, we show that P is completely positive. Suppose that $\tilde{x} \in \mathcal{M} \otimes M_{n\times n}$, where $M_{n\times n}$ is the algebra of $n \times n$ matrices, and $\tilde{x} \geqslant 0$. Then $(P \otimes \mathrm{id})(\tilde{x})$ is Hermitian and so $(P \otimes \mathrm{id})(\tilde{x}) = \int \lambda \,\mathrm{d}E(\lambda)$. Let Tr_n denote the normalized trace on $M_{n\times n}$. Because $(P \otimes \mathrm{id})(\tilde{x}) \in \mathcal{M}_1 \otimes M_{n\times n}$ so $E(B) \in \mathcal{M}_1 \otimes M_{n\times n}$ for any $B \in \mathcal{B}(\mathbb{R})$. Therefore,

$$(\mathrm{Tr} \otimes \mathrm{Tr}_n)[E(B)(P^\perp \otimes \mathrm{id})(\tilde{x})] = 0 = (\mathrm{Tr} \otimes \mathrm{Tr}_n)[E(B)(\tilde{x} - (P \otimes \mathrm{id})(\tilde{x}))]$$

which implies that

$$(\mathrm{Tr} \otimes \mathrm{Tr}_n)[E(B)(P \otimes \mathrm{id})(\tilde{x})] \geqslant 0$$

and so

$$(\mathrm{Tr}\otimes\mathrm{Tr}_n)\int_B \lambda\,\mathrm{d}E(\lambda)=\int_B \lambda\,\mathrm{d}(\mathrm{Tr}\otimes\mathrm{Tr}_n)E(\lambda)\geqslant 0.$$

Putting $B=(-\infty,0)$ we obtain that $E((-\infty,0))=0$ and so $(P\otimes\mathrm{id})(\tilde{x})$ is positive. Hence, by definition, P is completely positive. Because $P(\mathbf{1})=\mathbf{1}$ hence P satisfies the Schwarz inequality and so $\|P\|_{\infty,\infty}=1$. Thus, by duality, $\|P\|_{1,1}=1$. Because P coincides with the dual operator to a projector $P:L^1(\mathcal{M})\to L^1(\mathcal{M})$ so P is normal. □

Corollary 9. *$P:\mathcal{M}\to\mathcal{M}_1$ is a* Tr*-compatible σ-weakly continuous conditional expectation onto the algebra $\mathcal{M}_1$.*

Let us now determine what type of algebra $\mathcal{M}_1$ can be. The following theorem shows that there are no limits in this matter.

Theorem 10. *For any von Neumann subalgebra $\mathcal{N}\subset\mathcal{M}$ with identity, there is an operator T on $\mathcal{M}$, which satisfies assumptions (a)–(c) and whose effective subalgebra $\mathcal{M}_1$ coincides with $\mathcal{N}$.*

Proof. Let E be a Tr-compatibile σ-weakly continuous conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Because $\mathbf{1}\in\mathcal{N}$ so $E(\mathbf{1})=\mathbf{1}$. Suppose that $T(x)=ax+(1-a)E(x)$, where $x\in\mathcal{M}$ and $a\in(0,1)$. Then, by definition, T is normal and $T(\mathbf{1})=\mathbf{1}$. Moreover, $\|T(x)\|_\infty\leqslant\|x\|_\infty$. On the other hand,

$$\|E(x)\|_1=\sup_{\|y\|_\infty=1}|\mathrm{Tr}\,yE(x)|.$$

Let Ω denote the unit vector in the Hilbert space $L^2(\mathcal{M})$ determined by the identity operator $\mathbf{1}$. If $Q:L^2(\mathcal{M})\to L^2(\mathcal{M})$ is the orthogonal projection onto $\overline{\mathcal{N}\Omega}$, then for any $x\in\mathcal{M}$ there is $Q(x\Omega)=E(x)\Omega$. Hence

$$\begin{aligned}\mathrm{Tr}\,yE(x)&=\langle y^*\Omega,E(x)\Omega\rangle_{L^2}=\langle y^*\Omega,Q(x\Omega)\rangle_{L^2}\\&=\langle Q(y^*\Omega),x\Omega\rangle_{L^2}=\langle E(y^*)\Omega,x\Omega\rangle_{L^2}=\mathrm{Tr}\,E(y)x\end{aligned}$$

and so

$$\|E(x)\|_1=\sup_{\|y\|_\infty=1}|\mathrm{Tr}\,E(y)x|\leqslant\sup_{\|y\|_\infty=1}\|E(y)\|_\infty\|x\|_1=\|x\|_1.$$

Therefore, condition (b) is satisfied. Next, we show that T is completely positive. Let $M_{n\times n}$ denote the algebra of $n\times n$ matrices and let Tr_n be the normalized trace on it. Let e_{ij}, $i,j\in\{1,2,\ldots,n\}$ be a standard basis in $M_{n\times n}$. Suppose now that $\tilde{x}\in\mathcal{M}\otimes M_{n\times n}$. Then, for any $\tilde{y}\in\mathcal{N}\otimes M_{n\times n}$ there is

$$\begin{aligned}\mathrm{Tr}\otimes\mathrm{Tr}_n(E\otimes\mathrm{id}(\tilde{x})\tilde{y})&=\mathrm{Tr}\otimes\mathrm{Tr}_n\left(\sum_{i,j,k}E(x_{ik})y_{kj}e_{ij}\right)\\&=\frac{1}{n}\sum_{ij}\mathrm{Tr}\,E(x_{ik})y_{kj}=\frac{1}{n}\sum_{ij}\mathrm{Tr}\,x_{ik}y_{ki}=\mathrm{Tr}\otimes\mathrm{Tr}_n(\tilde{x}\tilde{y}).\end{aligned}$$

Hence, if $\tilde{x}\geqslant 0$, then $E\otimes\mathrm{id}(\tilde{x})\geqslant 0$ too, and so condition (a) follows. Finally, we determine the effective subalgebra $\mathcal{M}_1$ for the operator T. Let $\overline{T}:L^1(\mathcal{M})\to L^1(\mathcal{M})$ be the extension of T and $\overline{T}^*:\mathcal{M}\to\mathcal{M}$ the dual operator. We show that $T=\overline{T}^*$. Indeed, for any $x,y\in\mathcal{M}$ there is

$$\mathrm{Tr}[\overline{T}^*(x)y]=\mathrm{Tr}[xT(y)]=\mathrm{Tr}[x(ay+(1-a)E(y))]=\mathrm{Tr}[T(x)y].$$

Because $\mathcal{M}$ is dense in $L^1(\mathcal{M})$ so $\overline{T}^*(x) = T(x)$ for all $x \in \mathcal{M}$. Therefore, the property

$$T^{*n}T^n x = T^n T^{*n} x = x \qquad \text{for all} \quad n \in \mathbb{N}$$

simplifies to $T^2(x) = x$. However, $T^2(x) = a^2x + (1-a^2)E(x)$ and the equation $T^2(x) = x$ holds true only when $x = E(x)$. Therefore, $\mathcal{M}_1 = \mathcal{N}$. □

Suppose now that the subalgebra $\mathcal{M}_1$ is a maximal Abelian subalgebra in $\mathcal{M}$. By the representation theorem [20], it is isomorphic to the algebra $L^\infty([0,1], \mathcal{B}, \mathrm{d}x)$, L^∞ in short, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of interval [0, 1]. By theorem 7, $T : L^\infty \to L^\infty$ is an automorphism such that

$$\int_0^1 (Tf)(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$$

for any $f \in L^\infty$. Because $([0,1], \mathcal{B}, \mathrm{d}x)$ is a Lebesgue space so there is a bijective map $S : [0,1] \to [0,1]$ such that both S and S^{-1} are measurable and measure preserving, and $T(\chi_A) = \chi_{S^{-1}(A)}$ where χ_A denotes the characteristic function of a Borel set A. Thus the following theorem holds true.

Theorem 11. *The action of the group $(T|_{L^\infty})^k$, k is an integer, is induced by a discrete time trajectory $x \to S^k x$, $x \in [0,1]$, that is $(T^k f)(x) = f(S^k x)$ for any $f \in L^\infty$.*

In this way, we enter the area of measure spaces representing classical systems and their reversible evolution.

4. Example and concluding remarks

The main result of the previous section may be formulated as follows. Suppose that the evolution of an infinite quantum spin system interacting with its environment is given by a dynamical semigroup $T_t, t \geqslant 0$, which is also contractive in the trace norm. Then a von Neumann algebra $\mathcal{M}$ of observables of the system decomposes onto a von Neumann subalgebra $\mathcal{M}_1$ and a Banach subspace $\mathcal{M}_2$, which are both T_t-invariant. The evolution restricted to the algebra $\mathcal{M}_1$ is given by a one parameter group of *-automorphisms, while all expectation values (with respect to all statistical states of the system) of any operator belonging to $\mathcal{M}_2$ vanish when $t \to \infty$. More precisely, for any normal state $\phi \in \mathcal{M}_*$ and any $A \in \mathcal{M}_2$ there is

$$\lim_{t\to\infty} \mathrm{Tr}(T_t A)\phi = 0.$$

Since every $x \in \mathcal{M}$ may be written as $x = x_1 + x_2$, where $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$, in a unique way, we may conclude that only operators from the algebra $\mathcal{M}_1$ are available for measurements for arbitrary long times. This justifies the name of algebra of effective observables, at least in the case when decoherence happens almost instantaneously. Let us comment on the decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ (called the isometric-sweeping decomposition) which is obviously related to the asymptotic properties of the semigroup T^n. Such properties for positive or completely positive semigroups having a faithful normal stationary state (or a faithful family of subinvariant normal states) have been extensively studied by many authors. For example, in [21] and [22] the problem of the approach to equilibrium was addressed. In [18, 19, 23–26] the existence of the mean ergodic projection on a von Neumann algebra $\mathcal{M}$ was considered. Such a projection being a conditional expectation onto the fixed point subalgebra $\mathcal{M}^T$ provides another decomposition, namely $\mathcal{M} = \mathcal{M}^T \oplus \mathcal{N}$ with the obvious inclusion $\mathcal{M}^T \subset \mathcal{M}_1$. However, the evolution restricted to $\mathcal{M}^T$ is trivial, while on $\mathcal{M}_1$ it is given by a group of automorphisms. Moreover, the restriction of the dynamics to $\mathcal{N}$ cannot be controlled in general. For a partial result in this direction see [27]. From this point of view, the isometric-sweeping decomposition is closer to the so-called Jacobs–deLeeuw–Glicksberg splitting which holds whenever the semigroup is relatively compact in the weak operator topology, see for example [28]. Since, by proposition 1, trace is a stationary state for T^n so the pre-dual semigroup T_*^n is relatively compact in the weak operator topology on $L(\mathcal{M}_*)$. Hence, there exists a projection P in $L(\mathcal{M}_*)$ onto the so-called reversible part of $\mathcal{M}_*$. Its dual P^* provides another decomposition of $\mathcal{M}$, say $\mathcal{M} = \mathcal{M}_r \oplus \mathcal{M}_0$, with a possibly non-trivial evolution on its reversible part. However, again $\mathcal{M}_r \subset \mathcal{M}_1$ and $\mathcal{M}_2 \subset \mathcal{M}_0$. Thus the decomposition presented in section 3 seems to be the most optimal with respect to the process of decoherence which selects the algebra of effective observables. It is also worth noting that the isometric-sweeping decomposition exists also on type I factors, even when the Jacobs–deLeeuw–Glicksberg splitting fails to hold true [8].

In the case when decoherence affects all but a maximal subset of mutually compatible observables, then $\mathcal{M}_1$ becomes a maximal Abelian subalgebra of $\mathcal{M}$. Generalizing theorem 11 to the continuous case, we obtain that evolution of the algebra $\mathcal{M}_1$ is induced by a flow on the interval [0, 1], i.e. a measurable map $S : \mathbb{R} \times [0, 1] \to [0, 1]$, such that $(T_t f)(a) = f(S_t a)$ for any $f \in L^\infty$. Hence, the map $a \to S_t a$ is nothing other than a trajectory of a classical system represented in the Hilbert space $L^2([0, 1], \mathcal{B}, da)$ by the algebra L^∞. In this section, we present an example of the reduced dynamics of the quantum spin system for which the effective subalgebra becomes commutative, and whose evolution is (up to isomorphism) given by a uniform rotation of a circle.

Example. The model is the following. The quantum system is a semi-infinite linear array of spin-$\frac{1}{2}$ particles, fixed at positions $k \in \mathbb{N}$. The quasi-local algebra $\mathcal{A}$ is the norm closure of the algebra $\mathcal{A}_0 = \bigcup \mathcal{A}_n$ of local observables. Here by $\mathcal{A}_n$ we denote the $2^n \times 2^n$ matrix algebra associated with the set $\Lambda_n = \{1, 2, \ldots, n\}$. On the algebra $\mathcal{A}$ there is a unique normalized tracial state $\omega = \mathrm{Tr}$. Let $\mathcal{M} = \pi_\omega(\mathcal{A})''$, where π_ω is the corresponding GNS representation. Clearly, $\mathcal{M}$ is a type II_1 factor acting in the Hilbert space $\mathcal{H}_\omega = L^2(\mathcal{M})$. Suppose now that our system is open and interacts with its environment. Then, the reduced dynamics (in the Markovian approximation) of the algebra $\mathcal{M}$ is given by a master equation

$$\dot{x} = L(x) = \mathrm{i}[H, x] + L_0(x)$$

where $\delta = \mathrm{i}[H, \cdot]$ is a closed derivation generating a σ-weakly continuous one parameter group of *-automorphisms of $\mathcal{M}$, and L_0 represents the dissipative part of the generator L. Such a decomposition of the evolution generator onto the Hamiltonian and dissipative parts is widely used in physical models, see for example [29], where this issue together with a number of limiting procedures leading to such Markovian master equations is discussed. Dynamical semigroups with such a form of their generators for infinite fermion systems have also been studied by Davies [30]. Let us now describe these operators explicitly.

4.1. Construction of L_0

Suppose that D is a subalgebra in $\mathcal{A}$ generated by the identity operator $\mathbf{1}$ and the Pauli matrices σ_k^3, $k \in \mathbb{N}$. Clearly, D is a maximal Abelian subalgebra in $\mathcal{A}$ [31]. By the Gelfand theorem, it is isomorphic to a C^*-algebra of continuous functions on a compact space. It is easy to note that $D = C(\mathcal{C})$, where $\mathcal{C}$ is the Cantor set with appropriate topology. Let us recall that any $\phi \in \mathcal{C}$ is represented by a sequence $(i_1, i_2, \ldots)$, $i_n \in \{0, 2\}$ for all $n \in \mathbb{N}$, i.e. $\phi = \sum_n \frac{i_n}{3^n}$. Points of the Cantor set correspond to pure states of the algebra D, i.e. if $\phi = (i_1, i_2, \ldots) \in \mathcal{C}$

and $f \in D$, then

$$\phi(f) = \lim_{n\to\infty} \mathrm{Tr}\left(P_{i_1\ldots i_n} f\right)$$

where $P_{i_1\cdots i_n} = P_{i_1} \otimes \cdots \otimes P_{i_n}$ and P_{i_k} are minimal, complementary projections in $D \cap \mathcal{A}_{\{k\}}$. Clearly, $P_{i_1\cdots i_n}$ are minimal in $D \cap \mathcal{A}_n$. The Cantor set is homeomorphic with Ω^∞; an infinite product (with the product topology) of a two-point space $\Omega = \{0, 1\}$. Suppose that μ_0 is a probability measure on Ω such that $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$, and let μ_0^∞ be the tensor product of this measure. By μ we denote the regular Borel measure on the Cantor set induced by the homeomorphism $\mathcal{C} \to \Omega^\infty$. Then, for any $f \in D$, $\mathrm{Tr}\, f = \int f(x)\,\mathrm{d}\mu(x)$, where by the same symbol f we denote an element in D and the corresponding function in $C(\mathcal{C})$. Therefore, $\pi_\omega(D)'' = L^\infty(\mathcal{C}, \mathrm{d}\mu) \subset \mathcal{M}$. Suppose that

$$f_0(\phi) = \sum_{n=1}^{\infty} \frac{(-1)^{i_n/2}}{2^n}$$

where $\phi = (i_1, i_2, \ldots) \in \mathcal{C}$. Then $f_0 \in D$ and so $\pi_\omega(f_0) \in \mathcal{M}$. To derive an explicit formula for L_0 we use the so-called projection technique. Let the environment of the system consist of a single quantum particle located on a real line (in general, an environment is also an infinite quantum system but this simplified situation is sufficient for our purpose). Hence $\mathcal{H}_E = L^2(\mathbb{R}, \mathrm{d}s)$ and $\mathcal{M}_E = B(\mathcal{H}_E)$, the algebra of all bounded operators on $\mathcal{H}_E$. Such a joint system (a semi-infinite linear array of spin-$\frac{1}{2}$ particles plus a moving quantum particle), although without tracing out the particle variables, was discussed by Bell [32] in connection with the reduction of the wavefunction problem. On the joint algebra $\mathcal{M} \otimes \mathcal{M}_E$ we consider a unitary evolution determined by an interacting Hamiltonian

$$\tilde{H} = \pi_\omega(f_0) \otimes \hat{p} = \pi_\omega\left(\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \sigma_k^3\right) \otimes \hat{p}$$

where $\hat{p}$ is the momentum operator in $\mathcal{H}_E$. The reduced dynamics of the algebra $\mathcal{M}$ is given by the composition of the unitary dynamics of the joint system and the conditional expectation, i.e. a norm-1 and normal projection, with respect to a reference state ω_E of the environment, $\Pi^{\omega_E} : \mathcal{M} \otimes \mathcal{M}_E \to \mathcal{M}$. Suppose that $\omega_E = |\psi\rangle\langle\psi|$, where

$$\psi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}ps}\,\mathrm{d}p}{\sqrt{\pi(1+p^2)}}.$$

The evolution of any $x \in \mathcal{M}$ is given by

$$T_t^0(x) = \Pi^{\omega_E}(\mathrm{e}^{\mathrm{i}t\tilde{H}}(x \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}}).$$

Theorem 12. *T_t^0 is a quantum dynamical semigroup on $\mathcal{M}$, which is also contractive in the trace norm. Its generator L_0 is a bounded operator on $\mathcal{M}$.*

Proof. See the appendix. □

4.2. Construction of δ

Suppose that $D_n = D \cap \mathcal{A}_n$, i.e. D_n is isomorphic to the algebra of $2^n \times 2^n$ diagonal matrices. Suppose that $\mathcal{U}_n$ consists of all unitary operators U contained in $\mathcal{A}_n$ such that $U^* D_n U = D_n$. Clearly, $\mathcal{U}_n$ is a group and the inclusion $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ holds true. Let $U\left(\frac{1}{2^n}\right) \in \mathcal{U}_n$ be defined by the following property,

$$U\left(\frac{1}{2^n}\right)^* dU\left(\frac{1}{2^n}\right) = (d_{2^n 2^n}, d_{11}, d_{22}, \ldots)$$

where $d = (d_{11}, \ldots, d_{2^n 2^n})$. Clearly, it generates a cyclic group Z_{2^n} of rank 2^n. Because $Z_{2^n} \subset Z_{2^{n+1}}$ so $Z_\infty = \bigcup_n Z_{2^n}$ is an Abelian group of unitary operators in $\mathcal{A}$. It is isomorphic to the group of dyadic numbers

$$\mathcal{D} = \left\{ \frac{k}{2^n} : n \in \mathbb{N}, 0 \leqslant k \leqslant 2^n - 1 \right\}$$

with addition modulo 1. More precisely, if $d \in \mathcal{D}$, i.e. $d = \frac{k}{2^n}$, then $U(d) = U\left(\frac{1}{2^n}\right)^k$. It implies that the map $d \in \alpha(d)$, where $\alpha(d)(x) = U(d)^* x U(d), x \in \mathcal{M}$, is a group homomorphism into the group of inner automorphisms of $\mathcal{M}$. Let $\mathrm{Aut}(\mathcal{M})$ denote the group of *-automorphisms of the algebra $\mathcal{M}$.

Theorem 13. *There is a σ-weakly continuous group homomorphism $\alpha : t \in \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, such that $\alpha(m) = \mathrm{id}$ for any integer m, and for any $d \in \mathcal{D}, \alpha(d) = U(d)^* \cdot U(d)$. Moreover, $\alpha(t)$ is spatial, i.e. $\alpha(t) = U(t) \cdot U(t)^*$ where $U(t)$ is a strongly continuous group of unitary operators in $\mathcal{H}_\omega$.*

Proof. See the appendix. □

Suppose that δ is the generator of $\alpha(t)$. Clearly, $\delta = \mathrm{i}[H, \cdot]$, where H is a self-adjoint operator in the Hilbert space $\mathcal{H}_\omega$, the generator of the group $U(t)$. It is worth pointing out that δ is not inner since H is not affiliated to $\mathcal{M}$. Because L_0 is bounded so the operator $L = \delta + L_0$ with $D(L) = D(\delta)$ generates a σ-weakly continuous semigroup T_t which satisfies the conditions (a)–(c) from section 3. Next, we determine the subalgebra of effective observables $\mathcal{M}_1$ for the semigroup T_t.

4.3. Description of the subalgebra $\mathcal{M}_1$

Theorem 14. $\mathcal{M}_1 = \pi_\omega(D)'' = L^\infty(\mathcal{C}, \mathrm{d}\mu)$.

Proof. See the appendix. □

Let us now describe the action of the semigroup T_t restricted to the subalgebra $\mathcal{M}_1$. First, we do it for dyadic numbers. Elements of the group Z_∞ induce the following homeomorphisms of the Cantor set. If $d \in \mathcal{D}$, then $U(d) = U\left(\frac{1}{2^n}\right)^k$ and $\gamma(d) : \mathcal{C} \to \mathcal{C}$ is given by $\gamma(d) = \gamma\left(\frac{1}{2^n}\right) \circ \cdots \circ \gamma\left(\frac{1}{2^n}\right)$, where

$$\gamma\left(\frac{1}{2^n}\right)(i_1, i_2, \ldots) = (i_1, i_2, \ldots) \hat{+} (0, 0, \ldots, 2, 0, \ldots) \qquad (i_1, i_2, \ldots) \in \mathcal{C}.$$

Number 2 appears in the nth position, and the sum $\hat{+}$ is defined as follows:

$$(2, 2, \ldots, 2, i_{n+1}, \ldots) \hat{+} (0, 0, \ldots, 2, 0, \ldots) = (0, 0, \ldots, 0, i_{n+1}, \ldots).$$

If there exist $i_{j_1} < \cdots < i_{j_k}$, $j_k \leqslant n$, equal to zero, then $i_{j_k} \to 2, i_l \to 0$ for $j_k < l \leqslant n$, and the other indices remain unchanged. It is worth noting that points of the Cantor set $(i_1, i_2, \ldots), i_j \in \{0, 2\}$ correspond to all possible configurations of the position (up and down) of the third component of spin particles of the system. The action of $\gamma(d)$ changes such a configuration in the way described above. Suppose that $S^1 = \{\mathrm{e}^{\mathrm{i}a}, a \in \mathbb{R}\}$ and let $\lambda : \mathcal{C} \to S^1$ be given by

$$\lambda(i_1, i_2, \ldots) = \exp\left(2\pi \mathrm{i} \sum_{n=1}^{\infty} \frac{i_n}{2^{n+1}}\right).$$

Clearly, the map $\lambda : \mathcal{C} \to S^1$ is continuous and onto. Moreover, it induces an isomorphism of the following algebras $\hat{\lambda} : L^\infty(S^1, \mathrm{d}a) \to L^\infty(\mathcal{C}, \mathrm{d}\mu)$, $\lambda(f)(i_1, i_2, \ldots) = f(\lambda(i_1, i_2, \ldots))$, $f \in L^\infty(S^1, \mathrm{d}a)$, where $\mathrm{d}a$ denotes the normalized Lebesgue measure on the Borel σ-algebra of the circle S^1. It is easy to check that the automorphism $\hat{\alpha}(t) = \hat{\lambda}^{-1}\alpha(t)\hat{\lambda}$ for $t \in \mathcal{D}$ is unitary and given by the following formula $(\hat{\alpha}(t)f)(\mathrm{e}^{\mathrm{i}a}) = f(\mathrm{e}^{\mathrm{i}(a+2\pi t)})$, where $f \in L^\infty(S^1, \mathrm{d}a)$. By continuity and periodicity, this formula holds true for arbitrary $t \in \mathbb{R}$. Hence $\hat{\alpha}(t)$ is a group of automorphisms of the algebra $L^\infty(S^1, \mathrm{d}a)$, which is induced by a continuous flow on the underlying configuration space $\mathrm{e}^{\mathrm{i}a} \to \mathrm{e}^{\mathrm{i}(a+2\pi t)}$, i.e. a uniform rotation of the circle. Clearly, the presented example is formal. In a more realistic situation, one would have to consider the coupling of the infinite spin system with another infinite quantum system such as, for example, a phonon field at a positive temperature. Since temperature representations of boson fields are known to be factors of type III, such a generalization of the spin-boson model to infinite number of spin particles is technically much more involved and it is still under consideration.

4.4. Concluding remarks

The problem of the transition from microscopic to macroscopic worlds is a fundamental one in the discussion of interpretation of quantum mechanics. In particular, the emergence of classical dynamics described by differential, and hence local, equations of motion from the evolution of delocalized quantum states is at the centre of this issue. It is believed that environmentally induced decoherence, which destroys the majority of quantum superpositions, is responsible for emergence of classical properties in quantum systems. Such effects should be the most transparent in quantum systems consisting of many particles [33]. In the above example, we showed that it is possible, at least on the mathematical level, to force the infinite open quantum spin system to obey classical dynamics. In other words, the effective observables of the system may be parametrized by a single collective variable (representing possible configurations of the third component of spins) with periodic evolution. It should be pointed out, however, that this analysis does not solve the problem of transition from a quantum to a classical description of Nature. We believe that the main achievement of decoherence lies not in deriving the laws of classical physics from quantum theory but in demonstrating that in specific circumstances, introduced by approximations and guesses, quantum systems may be effectively described as classical ones, even those with non-trivial dynamical features.

Acknowledgments

We would like to thank the referee for his comments. This work was supported by the KBN research grant no 5P03B 081 21.

Appendix

Proof of theorem 12. To prove the first statement, one has to check only the semigroup property $T_t^0 T_s^0 = T_{t+s}^0$, $t, s \geqslant 0$. Because T_t^0 is normal and contractive in the operator norm so it is enough to show that $T_t^0 T_s^0(\pi_\omega(A)) = T_{t+s}^0(\pi_\omega(A))$ for all local operators A. Let us define a sequence of step functions,

$$f_n(\phi) = f_n(i_1, \ldots, i_n) = \sum_{k=1}^{n} \frac{(-1)^{i_k/2}}{2^k}$$

where $\phi \in \mathcal{C}$. It is clear that $f_n \in D \cap \mathcal{A}_n$, and that $f_n \to f_0$ in the sup norm. Hence $\mathrm{e}^{\mathrm{i}t\tilde{H}_n} \to \mathrm{e}^{\mathrm{i}t\tilde{H}}$ in the strong operator topology, where $\tilde{H}_n = \pi_\omega(f_n) \otimes \hat{p}$. First, we show that

$$\mathrm{e}^{\mathrm{i}t\tilde{H}}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}} = \mathrm{e}^{\mathrm{i}t\tilde{H}_n}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}_n}$$

for any $A \in \mathcal{A}_n$. Since

$$f_n = \sum_{i_1,\cdots,i_n} f_n(i_1,\ldots,i_n) P_{i_1\cdots i_n}$$

where $P_{i_1\cdots i_n}$ are minimal projections in $D \cap \mathcal{A}_n$ so, for any $k \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t\tilde{H}_{n+k}}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}_{n+k}} &= \sum_{\substack{i_1,\ldots,i_{n+k}\\ j_1,\ldots,j_{n+k}}} \pi_\omega\left(P_{i_1\cdots i_{n+k}} A P_{j_1\cdots j_{n+k}}\right) \otimes \mathrm{e}^{\mathrm{i}t(f_{n+k}(i_1,\ldots,i_{n+k}) - f_{n+k}(j_1,\ldots,j_{n+k}))\hat{p}} \\
&= \sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} \pi_\omega\left(P_{i_1\cdots i_n} A P_{j_i\cdots j_n}\right) \otimes \mathrm{e}^{\mathrm{i}t(f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n))\hat{p}} \\
&= \mathrm{e}^{\mathrm{i}t\tilde{H}_n}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}_n}.
\end{aligned}$$

Because

$$\lim_{k\to\infty} \mathrm{e}^{\mathrm{i}t\tilde{H}_{n+k}}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}_{n+k}} = \mathrm{e}^{\mathrm{i}t\tilde{H}}(\pi_\omega(A) \otimes \mathbf{1}_E)\,\mathrm{e}^{-\mathrm{i}t\tilde{H}}$$

in the strong operator topology so the required property follows. Using the above formula, we may calculate $T_t^0(\pi_\omega(A))$ for any local operator A. Since $A \in \mathcal{A}_0$ so there exists $n \in \mathbb{N}$ such that $A \in \mathcal{A}_n$. Thus

$$\begin{aligned}
T_t^0(\pi_\omega(A)) &= \Pi^{\omega_E}\left(\sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} \pi_\omega\left(P_{i_1\cdots i_n} A P_{j_1\cdots j_n}\right) \otimes \mathrm{e}^{\mathrm{i}t(f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n))\hat{p}}\right) \\
&= \sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} \pi_\omega\left(P_{i_1\cdots i_n} A P_{j_1\cdots j_n}\right) \langle \psi, \mathrm{e}^{\mathrm{i}t(f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n))\hat{p}}\psi\rangle.
\end{aligned}$$

However, for any $t \geqslant 0$,

$$\begin{aligned}
\langle \psi, \mathrm{e}^{\mathrm{i}t(f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n))\hat{p}}\psi\rangle &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t(f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n))p} \frac{\mathrm{d}p}{1+p^2} \\
&= \mathrm{e}^{-t|f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n)|}.
\end{aligned}$$

Hence

$$T_t^0(\pi_\omega(A)) = \sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} \mathrm{e}^{-t|f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n)|}\pi_\omega\left(P_{i_1\cdots i_n} A P_{j_1\cdots j_n}\right)$$

and so the semigroup property follows by direct calculations. Let L_0 denote the generator of this semigroup. If $A \in \mathcal{A}_n$, then $L_0(\pi_\omega(A)) = \pi_\omega(L_n \circ A)$, where $\circ$ is the Hadamard (entrywise) product and L_n is a $2^n \times 2^n$ matrix whose coefficients are given by the following formula:

$$(L_n)_{i_1\cdots i_n, j_1\cdots j_n} = -|f_n(i_1,\ldots,i_n) - f_n(j_1,\ldots,j_n)|.$$

Finally, we show that L_0 is a bounded operator on $\mathcal{M}$. To this end, we need a lemma.

Lemma. $L_n : \mathcal{A} \to \mathcal{A}$, $L_n(A) = L_n \circ A$, *is bounded with* $\|L_n\| \leqslant 4$.

Proof of the lemma. Suppose that B is a $2^n \times 2^n$ matrix. By K_B we denote the $\|\cdot\|_{\infty,\infty}$ norm of the operator B acting on the algebra $M_{2^n\times 2^n}$ as $A \to B \circ A$, i.e. $K_B = \sup\{\|B \circ A\|_\infty : \|A\|_\infty = 1\}$. Because

$$L_n = -\frac{1}{2^{n-1}}\begin{pmatrix} 0 & 1 & \dots & 2^n-1 \\ 1 & 0 & \dots & 2^n-2 \\ \dots & \dots & \dots & \dots \\ 2^n-1 & 2^n-2 & \dots & 0 \end{pmatrix}$$

so $L_n = L_n^+ + L_n^-$, where $L_n^+ (L_n^-)$ denotes respectively the upper triangular (lower triangular) part of the matrix L_n. Suppose that

$$(R_n)_{ij} = \begin{cases} 0 & i \leqslant j \\ 1 & i > j \end{cases} \qquad (S_n)_{ij} = \begin{cases} 0 & i > j \\ 1 & i \leqslant j \end{cases}$$

where $i, j \in \{1, 2, \dots, 2^n\}$. Then $S_n^* R_n = 2^{n-1} L_n^-$. Since

$$K_B = \min\{(c(S)c(R) : S^*R = B\}$$

see, for example, [34] for notation, so

$$2^{n-1} K_{L_n^-} \leqslant c(S_n)c(R_n) \leqslant \sqrt{2^n}\sqrt{2^n-1} \leqslant 2^n$$

what implies that $K_{L_n^-} \leqslant 2$. In the same way we may show that $K_{L_n^+} \leqslant 2$. Hence $K_{L_n} \leqslant 4$, which ends the proof of the lemma. □

Because T_t^0 is normal so there exists a strongly continuous semigroup T_{t*}^0 on $\mathcal{M}_*$ such that $T_t^0 = \left(T_{t*}^0\right)^*$. Let L_{0*} be its generator. Since T_{t*}^0 and T_t^0 coincide on $\pi_\omega(\mathcal{A}_0)$ so $\pi_\omega(\mathcal{A}_0) \subset D(L_{0*})$ and $L_{0*}(\pi_\omega(A)) = L_0(\pi_\omega(A))$ for $A \in \mathcal{A}_0$. Moreover, since the matrix L_n is symmetric and $\omega = \mathrm{Tr}$ so

$$\|L_{0*}(\pi_\omega(A))\|_1 = \|\pi_\omega(L_n \circ A)\|_1 = \|L_n \circ A\|_1 \leqslant K_{L_n}\|A\|_1 \leqslant 4\|\pi_\omega(A)\|_1$$

for any $A \in \mathcal{A}_0$. Hence L_{0*} is bounded on a norm dense subset in $\mathcal{M}_*$. However, L_{0*} is closed. Thus it is bounded on $\mathcal{M}_*$ and so L_0 is bounded on $\mathcal{M}$. □

Proof of theorem 13. First, we show the following lemma.

Lemma. *For any $A, B, C \in \mathcal{A}_0$ there exists a continuous function $f_{B,C}^A : [0, 1] \to \mathbf{C}$ such that $f_{B,C}^A(0) = f_{B,C}^A(1)$ and*

$$f_{B,C}^A(d) = \langle \pi_\omega(B)\Omega, \alpha(d)(\pi_\omega(A))\pi_\omega(C)\Omega\rangle$$

for all $d \in \mathcal{D}$.

Proof. Because Ω is a trace vector so we may assume that $C = \mathbf{1}$. For $d \in \mathcal{D}$ we put

$$f_{B,\mathbf{1}}^A(d) = \langle \pi_\omega(B)\Omega, \alpha(d)(\pi_\omega(A))\Omega\rangle.$$

Suppose now that $t \in [0, 1]\backslash\mathcal{D}$. Let us define a non-decreasing sequence of dyadic numbers d_k by induction,

$$d_1 = \begin{cases} 0 & t \leqslant \frac{1}{2} \\ \frac{1}{2} & t > \frac{1}{2} \end{cases} \qquad d_{k+1} = \begin{cases} d_k & t \leqslant d_k + \frac{1}{2^{k+1}} \\ d_k + \frac{1}{2^{k+1}} & t > d_k + \frac{1}{2^{k+1}} \end{cases}.$$

Clearly, $d_k < t$ and $d_k \to t$. First, we show that $\left(f_{B,\mathbf{1}}^A(d_k)\right)$ is a Cauchy sequence. Because $A, B \in \mathcal{A}_0$ so there exists $m \in \mathbb{N}$ such that $A, B \in \mathcal{A}_m$. For arbitrary $k, l \in \mathbb{N}$ there is

$$d_{m+k} = \frac{n_m}{2^m} + \frac{1}{2^m}\frac{n_k}{2^k} \qquad d_{m+k+l} = \frac{n_m}{2^m} + \frac{1}{2^m}\frac{n_k}{2^k} + \frac{1}{2^{m+k}}\frac{n_l}{2^l}$$

where $n_m \in \{0, 1, \ldots, 2^m - 1\}$ and so on. Since $\alpha\left(\frac{n_m}{2^m}\right)A \in \mathcal{A}_m$ so

$$\alpha\left(\frac{n_m}{2^m}\right)A = \sum_{i,j}^{2^m} a_{ij} E_{i,j}^{(m)}$$

where $\left\{E_{i,j}^{(m)}\right\}$, $i, j \in \{1, 2, \ldots, 2^m\}$, form the standard (linear) basis in $\mathcal{A}_m$. Finally, we define two local operators

$$\Lambda = \sum_{i,j}^{2^m} \lambda_{ij} E_{i,j}^{(m)}$$

where $\lambda_{ij} = a_{(i-1)(j-1)} - a_{ij}$ ($a_{0j} = a_{2^m j}$, $a_{i0} = a_{i2^m}$, $a_{00} = a_{2^m 2^m}$), and

$$F_{ij} = \sum_{p=2^l n_k+1}^{2^l n_k + n_l} E^{(m+k+l)}_{2^{k+l}(i-1)+p,\,2^{k+l}(j-1)+p}$$

if $n_l \geqslant 1$. For $n_l = 0$ we put $F_{ij} = 0$. Then

$$\alpha(d_{m+k+l})A - \alpha(d_{m+k})A = \sum_{i,j}^{2^m} \lambda_{ij} F_{ij}$$

and so

$$\mathrm{Tr}\,\pi_\omega(B)^*[\pi_\omega(\alpha(d_{m+k+l})A) - \pi_\omega(\alpha(d_{m+k})A)] = \frac{n_l}{2^{k+l}}\mathrm{Tr}\,\pi_\omega(B)^*\pi_\omega(\Lambda).$$

Because $n_l < 2^l$ so

$$\left|f^A_{B,\mathbf{1}}(d_{m+k+l}) - f^A_{B,\mathbf{1}}(d_{m+k})\right| \leqslant 2^{-k}\|B\|_\infty \cdot \|\Lambda\|_\infty.$$

However, $\|\Lambda\|_\infty$ depends only on $\alpha\left(\frac{n_m}{2^m}\right)A \in \mathcal{A}_m$, which implies that $\left(f^A_{B,\mathbf{1}}(d_k)\right)$ is a Cauchy sequence. Hence we may define

$$f^A_{B,\mathbf{1}}(t) = \lim_{k\to\infty} f^A_{B,\mathbf{1}}(d_k).$$

It is easy to check that if d'_k is an arbitrary sequence of dyadic numbers such that $d'_k < t$ and $d'_k \to t$, then

$$\lim_{k\to\infty} f^A_{B,\mathbf{1}}(d'_k) = \lim_{k\to\infty} f^A_{B,\mathbf{1}}(d_k).$$

Therefore, $f^A_{B,\mathbf{1}}(t)$ is well defined for all $t \in [0, 1]$. Since for any $d \in \mathcal{D}\backslash\{0\}$ there is

$$\lim_{k\to\infty} f^A_{B,\mathbf{1}}(d_k) = f^A_{B,\mathbf{1}}(d)$$

if $d_k < d$, $d_k \to d$ so, by definition, the function $f^A_{B,\mathbf{1}}$ is left-continuous. In order to show the right continuity of $f^A_{B,\mathbf{1}}$ it is sufficient to check that $\overline{f^B_{A,\mathbf{1}}(1-t)} = f^A_{B,\mathbf{1}}(t)$ or, equivalently, that for $t \in (0, 1)$,

$$\lim_{k\to\infty} f^A_{B,\mathbf{1}}(d_k) = \lim_{k\to\infty} \overline{f^B_{A,\mathbf{1}}(d'_k)}$$

where $d_k < t$, $d_k \to t$, and $d'_k < 1 - t$, $d'_k \to 1 - t$. Suppose that $\epsilon > 0$. Because

$$\begin{aligned}\left|\overline{f^B_{A,\mathbf{1}}(d'_k)} - f^A_{B,\mathbf{1}}(d_k)\right| &\leqslant \|A\|_\infty \left(2 f^B_{B,\mathbf{1}}(1) - f^B_{B,\mathbf{1}}(b_k) - \overline{f^B_{B,\mathbf{1}}(b_k)}\right)\\ &= 2\|A\|_\infty \,\mathrm{Re}\left(f^B_{B,\mathbf{1}}(1) - f^B_{B,\mathbf{1}}(b_k)\right)\end{aligned}$$

where $b_k = d_k + d'_k$, so, by the left continuity of $f^B_{B,\mathbf{1}}$ in point 1, $\left|\overline{f^B_{A,\mathbf{1}}(d'_k)} - f^A_{B,\mathbf{1}}(d_k)\right| < \epsilon$ for large $k \in \mathbb{N}$. Hence $\overline{f^B_{A,\mathbf{1}}(1-t)} = f^A_{B,\mathbf{1}}(t)$ and so $f^A_{B,\mathbf{1}}$ is continuous on the interval

$(0, 1)$, left-continuous in point 1, and right-continuous in point 0, which ends the proof of the lemma. □

Suppose now that $d_n \in \mathcal{D}$ and $d_n \to t \in [0, 1]$. Because for any $\Psi \in \mathcal{H}_\omega$ and $\epsilon > 0$ there exists $B \in \mathcal{A}_0$ such that $\|\pi_\omega(B)\Omega\|_2 \leqslant \|\Psi\|_2$ and $\|\Psi - \pi_\omega(B)\Omega\|_2 < \epsilon$ so

$$|\langle\Psi, (\alpha(d_n) - \alpha(d_m))(\pi_\omega(A))\Psi\rangle| \leqslant 2\epsilon\|A\|_\infty(\|\Psi\|_2 + \|\pi_\omega(B)\Omega\|_2) + |\langle\pi_\omega(B)\Omega, (\alpha(d_n) - \alpha(d_m))(\pi_\omega(A))\pi_\omega(B)\Omega\rangle| \leqslant 4\epsilon\|A\|_\infty\|\Psi\|_2 + \left|f^A_{B,B}(d_n) - f^A_{B,B}(d_m)\right|.$$

Hence, by the polarization formula, $\alpha(d_n)(\pi_\omega(A))$ converges weakly to some operator, say $\alpha(t)(\pi_\omega(A)) \in \mathcal{M}$. In particular, the function $f^A_{x,y} : [0, 1] \to \mathbf{C}$, $A \in \mathcal{A}_0$, $x, y \in \mathcal{M}$, is well defined and continuous. If $x \in \mathcal{M}$, $x \geqslant 0$, and $B, C \in \mathcal{A}_0$, then

$$\langle\pi_\omega(B)\Omega, (\alpha(d_n) - \alpha(d_m))(x)\pi_\omega(C)\Omega\rangle = f^{CB^*}_{x^{1/2},x^{1/2}}(1 - d_n) - f^{CB^*}_{x^{1/2},x^{1/2}}(1 - d_m).$$

Hence, by linearity, $\alpha(d_n)(x)$ converges weakly in $\mathcal{M}$ for all $x \in \mathcal{M}$. Its limit we denote by $\alpha(t)(x)$. Because

$$\|\alpha(t)(x)\Psi\|_2^2 = \langle\Psi, \alpha(t)(x^*x)\Psi\rangle = \lim_{n\to\infty}\langle\Psi, \alpha(d_n)(x^*x)\Psi\rangle \leqslant \|x\|_\infty^2\|\Psi\|_2^2$$

hence $\|\alpha(t)(x)\|_\infty \leqslant \|x\|_\infty$ for all $t \in [0, 1]$. It is also clear that if $d_n \to 1$, then $\alpha(d_n)(x)$ tends weakly to x. Thus for any $t \in [0, 1]$ the map $\alpha(t) : \mathcal{M} \to \mathcal{M}$ is well defined, linear, preserving the *-operation, and such that $\alpha(0) = \alpha(1) = \mathrm{id}$. It is easy to check that for any $d \in \mathcal{D}$ and $t \in [0, 1]$ there exists

$$\alpha((t + d)_{\mathrm{mod}\,1}) = \alpha(t) \circ \alpha(d) = \alpha(d) \circ \alpha(t)$$

and, in consequence, $\alpha((t_1 + t_2)_{\mathrm{mod}\,1}) = \alpha(t_1) \circ \alpha(t_2)$ for all $t_1, t_2 \in [0, 1]$. Moreover, since $\|\alpha(t)(x)\|_\infty \leqslant \|x\|_\infty$ so for any $x \in \mathcal{M}$, $t \to \alpha(t)(x)$ is σ-weakly continuous. Using the property $\alpha(0) = \alpha(1) = \mathrm{id}$ we extend it to a periodic and σ-weakly continuous map defined for all real numbers. Because $\alpha(t)(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{R}$ so, by corollary 3.2.13 in [13], $\alpha(t)$ is an *-automorphism. Finally, we show that $\alpha(t)$ is spatial. Let $\Psi = \mathbf{1}$ be the cyclic and separating vector in $\mathcal{H}_\omega$. For any $x \in \mathcal{M}$ we define $U_t(x\Psi) = (\alpha(t)x)\Psi$. Since U_t is densely defined and bounded we extend it onto the whole $\mathcal{H}_\omega$. Because

$$\mathrm{Tr}\,\alpha(t)x = \lim_{n\to\infty}\mathrm{Tr}\,\alpha(d_n)x = \mathrm{Tr}\,x$$

so U_t is a one parameter group of unitary operators. It is also clear that $U_t\Psi = U_t^*\Psi = \Psi$. Hence, for all $x, y \in \mathcal{M}$,

$$U_t x U_t^* y\Psi = U_t(x\alpha(-t)(y)\Psi) = (\alpha(t)x)y\Psi$$

and so $U_t x U_t^* = \alpha(t)x$. □

Proof of theorem 14. From construction of the operator L_0 we infer that $\pi_\omega(D)'' \subset \ker L_0$. Since $\mathrm{e}^{t\delta} : \pi_\omega(D)'' \to \pi_\omega(D)''$ so, by the Trotter product formula, $\pi_\omega(D)'' \subset \mathcal{M}_1$. Suppose now that $x \in \ker L_0$ and $x \notin \pi_\omega(D)''$, i.e. $P(x) \neq x$, where $P : \mathcal{M} \to \pi_\omega(D)''$ is a conditional expectation onto the maximal Abelian subalgebra $\pi_\omega(D)''$ [31]. Let $y = x - P(x)$. Then $y \in \mathcal{M}_1$ and $y \neq 0$. We may assume that $\|y\|_2 = 1$. Let us take a sequence (A_n), $A_n \in \mathcal{A}_n$, such that $\pi_\omega(A_n) \to x$ in $L^2(\mathcal{M})$. Then $B_n = \pi_\omega(A_n) - P(\pi_\omega(A_n)) \in \pi_\omega(\mathcal{A}_n)$ and $B_n \to y$. Hence, there exists $n_0 \in \mathbb{N}$ such that $\left\|y - B_{n_0}\right\|_2 < \frac{1}{4}$. Because $\left\|\mathrm{e}^{tL_0}B_{n_0}\right\|_2 \to 0$, when $t \to \infty$, so there exists $t_0 > 0$ such that $\left\|\mathrm{e}^{t_0L_0}B_{n_0}\right\|_2 < \frac{1}{4}$. Thus

$$1 = \|\mathrm{e}^{t_0L_0}y\|_2 \leqslant \left\|\mathrm{e}^{t_0L_0}\left(y - B_{n_0}\right)\right\|_2 + \left\|\mathrm{e}^{t_0L_0}B_{n_0}\right\|_2 < \tfrac{1}{2}$$

the contradiction. Therefore, $\ker L_0 = \pi_\omega(D)''$. Finally, we show that $\mathcal{M}_1 \subset \ker L_0$. By definition, $L = \delta + L_0$ with $D(L) = D(\delta)$ and $L^* = -\delta + L_0$ with $D(L^*) = D(\delta)$, are generators of $T_t : \mathcal{M} \to \mathcal{M}$ and $T_t^* : \mathcal{M} \to \mathcal{M}$ respectively. Because the closure $\overline{L + L^*} = 2L_0$ is a generator of a contractive semigroup so, by the Trotter formula,

$$\mathrm{e}^{2tL_0} = \lim_{n\to\infty} (T_{t/n} T^*_{t/n})^n x$$

for all $x \in \mathcal{M}$. For $x \in \mathcal{M}_1$ we have $T_t T_t^* x = x$ for all $t \geqslant 0$, and hence $x \in \ker L_0$. □

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